About Increments of Additive Functionals of Diffusion Processes

SERGIO ALVAREZ-ANDRADE
Université de Technologie de Compiègne, FRANCE

ABSTRACT. We study the increments of additive functionals of diffusion processes by using strong approximations and some properties of the local times.

Keywords and phrases. Additive functionals of diffusion; Brownian motion; Invariance principles; Processes with independent and stationary increments.


1. Introduction

Let $X = \{X_t : t \geq 0\}$ be a recurrent one-dimensional diffusion living on an interval $I \subset \mathbb{R}$, with $X_0 = 0$. Csáki and Salminen in [3], established strong approximations results for additive functionals of the form

$$Z_t = \int_0^t f(X_s)ds = \int_I f(x)L_t^x m(dx),$$

(1)

where $f(x)$, $x \in I$ is a locally integrable real valued function with the property $\int_I |f(x)|m(dx) \leq \infty$, $L_t^x$ is the local time of $X_t$ at every point $x \in I$ and $m$ is the speed measure of $X$.

Our aim is to establish the asymptotic behavior of the increments of additive functionals defined as in (1) by using strong approximations results established in [3]. Our arguments are based on well known results for the increments of Brownian motion and those of local times.

In the sequel we consider $\log t = \log(\sup(t, e))$. 

87
Let $A_u = A_u^0$ be the right continuous inverse of $L_t = L_t^0$ and $S$ the scale function of $X$. If we put $t = A_u$ in (1), then we have

$$Z_{A_u} = \int_0^{A_u} f(X_s)ds = \int_0^t f(x)L_t^{A_u}m(dx),$$

and $\{Z_{A_u}, u \geq 0\}$ is a process with independent and stationary increments (see, Csáki and al., (1992), Csáki and Csörgö (1995) and Csáki and Salminen (1996)). Moreover, we have

$$EZ_{A_u} = u\bar{f},$$

where

$$\bar{f} = \int f(x)m(dx),$$

and

$$VarZ_{A_u} = u\sigma^2,$$

where

$$\sigma^2 = 2\iint_{I \times I \cap \{xy > 0\}} f(x)f(y)\min(|S(x)|, |S(y)|)m(dx)m(dy).$$

Two cases was considered in [3]: firstly, the case of positive recurrence (i.e. $\mu = EA_1 = m\{I\} < \infty$); secondly the case of null recurrence (i.e. $m\{I\} = \infty$).

The following strong approximation result was established in the case of positive recurrence.

**Theorem 1.** Assume that

$$E(A_1)^q < \infty \text{ for some } 1 < q \leq 2.$$  \hspace{1cm} (4)

(i) If

$$E\left(\int_0^{A_1} |f(X_s)|ds\right)^{2+\delta} < \infty,$$

for some $\delta > 0$, then on a suitable probability space one can construct a diffusion process $X_t$ and a standard Brownian motion $W(t)$ such that

$$|Z_t - \bar{f}L_t^0 - \frac{\sigma}{\sqrt{\mu}}W(t)| = O\left(t^\beta \log t\right) \text{ a.s.},$$

as $t \to \infty$, where

$$\beta = \max\left(\frac{1}{2+\delta}, \frac{1}{2q}\right),$$

and $\sigma^2$ is defined as in (3).
(ii) If
\[
E \left( \int_0^{A_1} |f(X_s) - \frac{\bar{f}}{\mu}| ds \right)^{2+\delta} < \infty, \tag{7}
\]
for some \( \delta > 0 \), then on a suitable probability space one can construct a diffusion process \( X_t \) and a standard Brownian motion \( W(t) \) such that
\[
|Z_t - \frac{\bar{f}}{\mu} - \sigma \sqrt{\mu} W(t)| = O \left( t^\beta \log t \right) \text{ a.s.},
\]
as \( t \to \infty \), where \( \beta \) is defined by (6) and
\[
\sigma^2 = 2 \int_{I \times I \setminus \{xy>0\}} (f(x) - \bar{f})(f(y) - \bar{f}) \min(|S(x)|, |S(y)|) m(dx)m(dy).
\]

In the case of null recurrence (i.e. \( m(I) \) is infinite) under assumption that on a suitable probability space one can construct a diffusion process \( X \) and a stable process \( T_u \) of order \( \alpha \) (cf. Samorodnitski and Taqqu (1994)) such that
\[
|A_u - T_u| = O(u^k) \text{ a.s.,} \tag{8}
\]
as \( u \to \infty \) for some \( 0 \leq k < 1/\alpha \) (see, [1]), the following result was established in [3].

**Theorem 2.** Assume that \( X \) is a null recurrent diffusion process on an interval \( I, 0 \in I \) with local time \( L^*_t \) and such that (7) and (8) are both satisfies. Then on a suitable probability space one can construct a diffusion process \( X_t \) and a standard Brownian motion \( W(u) \) and a non-decreasing stable process \( T_u \) of order \( \alpha \), such that \( W \) and \( T \) are independent and for \( \epsilon > 0 \) small enough, we have
\[
|Z_t - \bar{f}L^*_t - \sigma W(V_t)| = O \left( t^{\alpha/2-\epsilon} \right) \text{ a.s.},
\]
as \( t \to \infty \), where \( V_t \) is the (continuous) inverse of \( T_u \) and \( Z_t \), \( \bar{f} \) and \( \sigma \) are defined by (1), (2) and (3) respectively.

2. **Main results**

**Proposition 1.** Under conditions i) and ii) of Th.1, let \( h_T \) be a real function satisfying \( h_T \to +\infty \), \( h_T/T \) is non-increasing and \( \log(T/h_T) |\log T| = \infty \) as \( T \to \infty \). Then we have
\[
\sup_{0 \leq t \leq T-h_T} (Z_{t+h_T} - Z_t) = O(d_1(T)) \text{ a.s.},
\]
where
\[
d_1(T) = \sup(h_T, T^{\beta} \log T),
\]
under i) and
\[
d_2(T) = h_T/\mu + T^{\beta} \log T,
\]
under ii).
Proposition 2. Under conditions of Theorem 2, let $h_T = T^\beta$, we have

$$
\sup_{0 \leq t \leq T-h_T} (Z_{t+h_T} - Z_t) = O(T^{\beta+\epsilon}) \text{ a.s.}
$$

(9)

3. Proofs

Proof of Proposition 1. (i) Put $h = h_T$, by (5), we have

$$
|Z_{t+h} - Z_t| \leq \int |L_{t+h} - L_t| + \frac{\sigma}{\sqrt{t}} |W(t+h) - W(t)| + O((t + h)^{\beta \log (t + h)}) \text{ a.s.}
$$

(10)

It is clear that we only study the first two terms of the right hand side of (10). We begin by stating that

$$
|L_{t+h} - L_t| = O(d_0(t)) \text{ a.s.}
$$

Put $t+h = A_{u+h}$ in $|L_{t+h} - L_t|$. Under condition (4), we have

$$
A_u = \mu u + O(u^{1/q}(\log u)^{1/2}) \text{ a.s.},
$$

(11)

as $u \to \infty$ and consequently

$$
L_t = \frac{t}{\mu} + O(t^{1/q}(\log t)^{1/2}) \text{ a.s.,}
$$

as $t \to \infty$. By relations (10) and (11), we have as $u \to \infty$ that

$$
L_{A_{u+h}} = L_{A_u} \text{ a.s.}
$$

By Proposition (2.1) of Csáki and Salminen (1996), we have

$$
L_{A_u} = l_{a_u} \text{ a.s.,}
$$

where $l_t$ is the local time of the Wiener process $W(t)$ and $a_t$ their inverse right continuous function, then we can consider

$$
|L_{A_{u+h}} - L_{A_u}| = |l_{a_{u+h}} - l_{a_u}| \text{ a.s.}
$$

(12)

By using results for the increments of the local time process of the Wiener process (see, Csáki et al., (1992)), we have

$$
\limsup_{t \to \infty} \sup_{0 \leq s \leq t-h} \frac{d_0(t)}{|l_{A_{u+h}} - l_{A_u}|} = 1 \text{ a.s.,}
$$

where $d_0(t) = \sqrt{h}(\log(t/h)+2 \log \log t)$ and if $\log(t/h)/\log \log t = \infty$ then

$$
\limsup_{t \to \infty} \frac{\sup_{0 \leq s \leq t-h} |l_{A_{u+h}} - l_{A_u}|}{\sqrt{h \log(t/h)}} = 1 \text{ a.s.}
$$

If in the right term of (12) we put $t = a_u$, then we have

$$
|l_{t+h} - l_t| = O(d_0(t)) \text{ a.s.}
$$

(13)
This last result is sufficient for to get the announced result. For the remaining term, by Theorem 1.2.1 of Csörgő and Révész (1981), we have

$$|W(t + h) - W(t)| = O(d_0(t)) \ a.s., \quad (14)$$

and by condition $h/T$ is non-increasing when $T \to \infty$, we have that the $O(\ )$ term in (10) is an $O(T^\beta \log T)$. With this last result and by (13) and (14), we get the expected result. $\Box$ ✓

The proof of part ii) is close to the previous proof.

**Proof of Proposition 2.** In the same way as in Proposition 1, we have

$$|Z_t+ - Z_t| \leq \int |L_{t+h} - L_t| + \sigma |W(V_{t+h} - W(V_t)| + O(t^{\alpha/2 - \epsilon}) \ a.s. \quad (15)$$

For to study the right term of (15), we recall some results given in Csáki and Salminen (1996) : for $0 < \beta < 1$ and $\epsilon > 0$, the following relations are satisfied

$$\sup_{0 \leq s \leq t} (V_{s+t^\beta} - V_s) = O(t^{\alpha \beta + \epsilon}) \ a.s., \quad (16)$$

$$\sup_{0 \leq s \leq t} (L_{s+t^\beta} - L_s) = O(t^{\alpha \beta + \epsilon}) \ a.s., \quad (17)$$

$$|V_t - L_t| = O(t^{\alpha \beta + \epsilon}) \ a.s. \quad (14)$$

By (15), (16) and (17), we can deduce (9). $\Box$ ✓

**References**


(Recibido en noviembre de 2003)