Connected transversals to subnormal subgroups

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Abstract. Subnormal subgroups possessing connected transversals are briefly discussed.

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In [8] J.D.H. Smith introduced the notion of a stably nilpotent quasigroup, showing that a quasigroup $Q$ is stably nilpotent if and only if the inner permutation groups of $Q$ are subnormal in the multiplication group of $Q$. Generalizing this for abstract groups, we come by groups which are, in a certain sense, relatively nilpotent with respect to a subgroup. The present short note collects some basic information on such groups.

1. Preliminaries

1.1. Let $H$ be a subgroup of a group $G$. Then $L_G(H)$ denotes the core and $N_G(H)$ the normalizer of $H$ in $G$. Further, $N_G,0(H) = H$ and $N_G,n+1(H) = N_G(N_G,n(H))$ for every $n \geq 0$.

The subgroup $H$ is said to be subnormal of depth at most $n \geq 0$ in $G$ if there are subgroups $H_0, H_1, \ldots, H_n$ of $G$ such that $H_0 = H$, and $H_n = G$ and $H_i$ is normal in $H_{i+1}$ for every $0 \leq i \leq n - 1$.

1.2. Let $G$ be a group. For $n \geq 0$, $Z_n(G)$ denotes the $n$th member of the usual central series. That is, $Z_0(G) = 1$, and $Z_{n+1}(G)/Z_n(G) = Z(G/Z_n(G))$.

Now, let $H$ be a subgroup of $G$. We define two series of normal subgroups of $G$: $Z_{H,0}(G) = Z_{H,0}^*(G) = L_G(H)$, $Z_{H,n}(G) \subseteq Z_{H,n+1}^*(G)$ and $Z_{H,n+1}^*(G)/Z_{H,n}(G) = Z(G/Z_{H,n}(G))$, $Z_{H,n+1}(G) = L_G(H \cdot Z_{H,n+1}^*(G))$.

1.3 Remark. (i) A subgroup $H$ is subnormal of depth at most $n \geq 0$ in a group $G$, provided that $N_G,n(H) = G$. The converse is not true in general (see, e.g., 4.1).

(ii) If $G$ is a finite group, then subnormal subgroups form a sublattice in the lattice of all subgroups of $G$ (see, e.g., [6, Theorem 6.5]). This is not true in general ([7, §13.1, p.375]), albeit subnormal subgroups of arbitrary (i.e., even infinite) groups are closed under finite intersections.
2. Technical results

2.1 Lemma. Let \( H \) be a subgroup of a group \( G \). Then:

(i) \( L_G(H) = Z_{H,0}(G) \subseteq Z_{H,1}(G) \subseteq Z_{H,2}(G) \subseteq \ldots \);
(ii) \( L_G(H) = Z^*_{H,0}(G) \subseteq Z^*_{H,1}(G) \subseteq Z^*_{H,2}(G) \subseteq \ldots \);
(iii) \( Z_{H,n}(G) \subseteq Z^*_{H,n+1}(G) \subseteq Z_{H,n+1}(G) \subseteq Z^*_{H,n+2}(G) \subseteq \ldots \) for every \( n \geq 0 \);
(iv) \( Z_{H,n}(G) \subseteq L_G(N_{G,n}(H)) \) for every \( n \geq 0 \).

Proof: The first three assertions are clear from definition 1.2, (iv) is clear for \( n = 0 \), and we shall proceed further by induction.

Let \( f : G \rightarrow \overline{G} = G/Z_{H,n}(G) \), \( g : G \rightarrow \tilde{G} = G/L_G(N_{G,n}(H)) \) and \( h : \overline{G} \rightarrow \tilde{G} \) denote the natural projections, \( g = hf \). Then \( Z^*_{H,n+1}(G) = f^{-1}(Z(\tilde{G})) \subseteq g^{-1}(Z(\tilde{G})) = K \), \( HK \subseteq N_{G,n}(H)K \subseteq N_G(N_{G,n}(H)) = N_{G,n+1}(H) \) and \( Z_{H,n+1}(G) = L_G(H \cdot Z^*_{H,n+1}(G)) \subseteq L_G(HK) \subseteq L_G(N_{G,n+1}(H)). \)

2.2 Lemma. Let \( H \subseteq K \subseteq G \) be subgroups of a group \( G \). Then \( Z_{H,n}(G) \subseteq Z_{K,n}(G) \) and \( Z^*_{H,n}(G) \subseteq Z^*_{K,n}(G) \) for every \( n \geq 0 \).

Proof: By induction on \( n \) (see the proof of 2.1(iv)).

2.3 Lemma. Let \( H \) be a subgroup of a group \( G \). Then \( Z_n(G) \subseteq Z^*_{H,n}(G) \subseteq Z_{H,n}(G) \) for every \( n \geq 0 \).

Proof: Clearly, \( Z_n(G) \subseteq Z^*_{1,n}(G) \subseteq Z_{1,n}(G) \) and we can use 2.2.

2.4 Lemma. Let \( H \) be a subgroup of a group \( G \). Then:

(i) \( Z_{H,0}(G) = G \) iff \( H = G \);
(ii) \( Z_{H,1}(G) = G \) iff \( G' \subseteq H \);
(iii) \( Z_{H,n}(G) = G \) for \( n \geq 0 \) iff \( G = H \cdot Z^*_{H,n}(G) \);
(iv) if \( G \) is nilpotent of class at most \( n \geq 0 \), then \( Z_{H,n}(G) = G \);
(v) if \( Z_{H,n}(G) = G \) for \( n \geq 0 \), then \( N_{G,n}(H) = G \) (and hence \( H \) is subnormal of depth at most \( n \) in \( G \)).

Proof: The first assertions are easy, (iv) follows from 2.3, and (v) follows from 2.1(iv).

2.5 Lemma. Let \( H \) be a subgroup of a group \( G \) such the \( L_G(H) = 1 \). Then:

(i) \( Z^*_{H,1}(G) = Z(G) \) and \( Z_{H,1}(G) = L_G(HZ(G)) \);
(ii) \( Z_{H,1}(G) = G \) iff \( G \) is abelian;
(iii) \( Z_{H,2}(G) = G \) iff \( G' \subseteq HZ(G) \).

Proof: Obvious.
2.6 Lemma. Let $H$ be a subgroup of a group $G$. Then:

(i) $HZ_{H,n}(G) = HZ^*_{H,n}(G)$ for every $n \geq 0$;

(ii) if $K$ is a subgroup conjugate to $H$, then $Z_{H,n}(G) = Z_{K,n}(G)$ and $Z^*_{H,n} = Z^*_{K,n}(G)$ for every $n \geq 0$.

Proof: The assertions follow easily from definition 1.2. □

2.7 Proposition. Let $H$ be a subgroup of a group $G$. The following conditions are equivalent for $n \geq 1$:

(i) $Z^*_{H,n}(G) = G$;

(ii) $Z_{H,n}(G) = G$;

(iii) $HZ_{H,n}(G) = G$;

(iv) $HZ^*_{H,n}(G) = G$;

(v) $G' \subseteq Z_{H,n-1}(G)$;

(vi) $G' \subseteq HZ_{H,n-1}(G)$;

(vii) $G' \subseteq HZ^*_{H,n-1}(G)$.

Proof: (i) implies (ii) by 2.1 (iii); (ii) implies (iii) and (v) implies (vi) trivially; (iii) implies (iv) and (vi) implies (vii) by 2.6 (i).

We now show (iv) implies (v). Put $N = Z_{H,n-1}(G)$. We have $\overline{G} = G/N = HZ^*_{H,n}(G)/N = \overline{HZ(G)}$, and hence $(\overline{G})' \subseteq \overline{H}$, $G' \subseteq HN = HZ^*_{H,n-1}(G)$ and $N = L_G(HZ^*_{H,n-1}(G)) = HZ^*_{H,n-1}(G)$. Consequently $G' \subseteq N$. Finally, we show (vii) implies (i). Since $G' \subseteq HZ^*_{H,n-1}(G)$, we have $Z_{H,n-1}(G) = HZ_{H,n-1}(G), G' \subseteq Z_{H,n-1}(G)$ and $Z^*_{H,n}(G) = G$ (see 1.2). □

2.8. Let $H$ be a subgroup of a group $G$, $n \geq 0$, $N = Z_{H,n}(G), N^* = Z^*_{H,n}(G)$, $G = G/N$, and $H = HN/N \subseteq G$.

(i) $HN = HN^*, N = L_G(HN^*) = L_G(HN)$ and this implies that $L_G(H) = 1$ and $H = HN/N$.

(ii) $Z^*_{H,n+1}(G)/N = Z_{H,n+1}(G) = Z^*_{H,n+1}(G), Z_{H,n+1}(G) = L_G(H \cdot Z^*_{H,n+1}(G))$ and $Z_{H,n+1}(G)/N = L_G(HZ(G)) = Z_{H,n+1}(G)$.

(iii) $Z^*_{H,n+m}(G)/N = Z^*_{H,m}(G)$ and $Z_{H,n+m}(G)/N = Z^*_{H,m}(G)$ for every $m \geq 1$.

2.9. Let $H$ be a subgroup of a group $G$. Put $H_n = H \cap Z_{H,n}(G)$ for every $n \geq 0$. Then $L_G(H) = H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ and $H_n$ is normal in $G$.

2.10 Lemma. Let $H$ be a subgroup of a group $G$ such that $L_G(H) = 1$ and let $\alpha = [G : HZ(G)]$. Then:

(i) $Z_{H,1}(G) = L_G(HZ(G))$ can be embedded into the Cartesian product of $\alpha$ copies of $Z(G)$;

(ii) $Z_{H,1}(G)$ is an abelian group;

(iii) $H_1$ (see 2.9) can be embedded into the Cartesian product of $\alpha - 1$ copies of $Z(G)$ ($\alpha - 1 = \alpha$ for $\alpha$ infinite).
Proof: Put $N = Z_{H,1}(G)$. For every $x \in G$, $N = N^x = L_G(H^x \cdot Z(G))$, $H^x \cap Z(G) \subseteq L_G(H^x) = L_G(H) = 1$, $H^x \cdot Z(G)$ is the direct product of $H^x$ and $Z(G)$ and consequently the restriction $f_x$ of the natural projection $H^x \cdot Z(G) \to Z(G)$ to $N$ is a homomorphism of $N$ onto $Z(G)$ (we have $Z(G) \subseteq N$).

Now, let $A$ be a right transversal to $HZ(G)$ in $G$ such that $1 \in A$. Define a homomorphism $f : N \to \prod_{\alpha} Z(G)$ by $f(u) = \prod_{a \in A} f_a(u), u \in N$. If $u \in \text{Ker}(f)$, then $aua^{-1} \in H$ for every $a \in A$. Consequently, $u \in H$ and if $x \in G$, $x = zva$, $a \in A$, $v \in H$, $z \in Z(G)$, then $xux^{-1} = zvaua^{-1}v^{-1}z^{-1} = vaua^{-1}v^{-1} \in H$. Thus $u \in L_G(H) = 1$ and we have proved that $f$ is injective. Finally, for $g = \prod_{a \neq 1} f_a$ we get $\text{Ker}(g) \cap H = 1$, and hence $g|H_1$ is injective.

2.11 Proposition. Let $H$ be a subgroup of a group $G$ and let $\alpha_n = [G : H \cdot Z_{H,n+1}(G)]$ for every $n \geq 0$. Then $Z_{H,n+1}(G)/Z_{H,n}(G)$ is an abelian group which can be embedded into the Cartesian product of $\alpha_n$ copies of $Z(G/Z_{H,n}(G)) = Z_{H,n+1}^*(G)/Z_{H,n}(G)$.

Proof: The result follows by an easy combination of 2.10 and 2.8 (i),(ii).

2.12 Corollary. Let $H$ be a subgroup of a group $G$ such that $Z_{H,n}(G) = G$ for some $n \geq 0$. If $H$ is soluble of derived length $m \geq 0$, then $G$ is also soluble and its derived length is at most $n + m$.

2.13 Lemma. Let $H$ be a subgroup of a group $G$ such that $Z_{H,2}(G) = G$. Then $H \subseteq L_G(H)$.

Proof: By 2.10, $H/L_G(H)$ is abelian.

2.14 Proposition. Let $H$ be a subgroup of a finite group $G$ such that $[G : H]$ is a power of a prime $p$ and $L_G(H)$ is a $p$-group. Then $G = Z_{H,n}(G)$ for some $n \geq 0$ iff $G$ is a $p$-group.

Proof: If $G$ is a $p$-group, then $G$ is nilpotent and our result follows from 2.3. Now assume that $Z_{H,n}(G) = G$. We shall proceed by induction on $\text{card}(G)$. Further, considering the factor $G/L_G(H)$, we can restrict ourselves to the case $L_G(H) = 1$. Then $H \cap Z(G) = 1$, $[HZ(G) : H] = \text{card}(Z(G))$, and hence $Z(G)$ is a $p$-group. From this, $N = Z_{H,1}(G)$ is a $p$-group by 2.10 (i). Since $N \neq 1$ (otherwise $G = 1$), $G/N$ is a $p$-group by induction.

2.15. Let $H$ be a subgroup of a group $G$ such that $G/Z_{H,n}(G)$ is a two element group for some $n \geq 0$.

(i) If $n = 0$, then $G/L_G(H)$ is a two element group, which means that $H$ is normal and of index 2 in $G$.

(ii) Assume that $n \geq 1$. Clearly, $Z_{H,n+1}(G) = Z_{H,n+1}^*(G) = G$ and $G' \subseteq Z_{H,n}(G) = H \cdot Z_{H,n}^*(G)$. Put $N = Z_{H,n-1}(G)$, $G' = G/N$ and $\overline{G} = HN/N = HZ_{H,n-1}^*(G)/L_G(HZ_{H,n-1}^*(G))$. We have $L_{\overline{G}}(H) = 1$, $Z(\overline{G}) =$
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3.1. In this section, let $H$ be a subgroup of a group $G$ such that there exist $H$-connected transversals $A, B$ to $H$ in $G$ (i.e., $A, B$ are left transversals and $[A, B] \subseteq H$).

3.2 Lemma.

(i) $HZ_{H,n}(G) = HZ_{H,n}^*(G) = N_{G,n}(H)$ for every $n \geq 0$.

(ii) $Z_{H,n}(G) = L_G(N_{G,n}(H))$ for every $n \geq 0$.

Proof: This is clear for $n = 0$ and we shall proceed by induction on $n$.

Put $N = Z_{H,n}(G)$ and consider the factors $\overline{G} = G/N$ and $\overline{H} = HN/N$. Then $L_{\overline{G}}(\overline{H}) = 1$, and so $N_{\overline{G}}(\overline{H}) = \overline{HZ(\overline{G})}$ by [3, Proposition 2.7]. This implies that $N_G(HN) = HZ_{H,n+1}^*(G)$. However, $HN = N_{G,n}(H)$ by the induction and we have $N_{G,n+1}(H) = HZ_{H,n}^*(G) = HZ_{H,n}(G)$ (2.6 (ii)). The rest is clear. □

3.3 Proposition. The following conditions are equivalent for $n \geq 1$:

(i) $Z_{H,n}(G) = G$;

(ii) $HZ_{H,n-1}(G)$ is normal in $G$;

(iii) $H \subseteq Z_{H,n-1}(G)$;

(iv) $H_{n-1} = H$ (see 2.9);

(v) $H$ is subnormal of depth at most $n$ in $G$;

(vi) $N_{G,n}(H) = G$;

(vii) $N_G(H)$ is subnormal of depth at most $n - 1$ in $G$.

Proof: (i) implies (ii) by 2.7 (ii),(vi) (in fact, $G' \subseteq HZ_{H,n-1}(G)$); (ii) implies (iii), since $Z_{H,n-1}(G) = L_G(H Z_{H,n-1}(G))$; (iii) implies (iv) trivially; (iv) implies (ii), since $Z_{H,n-1}(G) = L_G(H Z_{H,n-1}(G))$; (i) implies (vi) by 2.1 (iv); (vi) implies (vii) and (vii) implies (v) trivially; (vi) implies (i) by 3.2 (ii).

We now show (ii) implies (i). The existence of $H$-connected transversals easily yields that $G' \subseteq HZ_{H,n-1}(G)$ (consider the factor $G/Z_{H,n-1}(G)$), and the result follows from 2.7.

We proceed by induction on $n$ to show (v) implies (vi). If $n = 1$, then $H$ is normal in $G$ and (vi) is clear. Let $n \geq 2$ and let $L_G(H) = 1$ (considering the factor $G/L_G(H)$, we can restrict ourselves to this case). There is a subgroup $K$ of $G$ such
that \( H \) is a normal subgroup of \( K \) and \( K \) is subnormal of depth at most \( n - 1 \) in \( G \).

Put \( L = L_G(K) \), \( \overline{G} = G/L \) and \( \overline{K} = K/L \). Then \( L_{\overline{G}}(\overline{K}) = 1 \) and \( \overline{K} \) is subnormal of depth at most \( n - 1 \) in \( \overline{G} \). Consequently, \( N_{G,n-1}(\overline{K}) = \overline{G} \) and \( N_{G,n-1}(K) = G \).

On the other hand, \( K \subseteq N_G(H) = HZ(G) \) ([3, Proposition 2.7]), and hence \( N_G(H) = KZ(G) \) is normal in \( N_G(K) \). We have proved that \( N_G(H) \) is subnormal of depth at most \( n - 1 \) in \( G \).

Using the induction hypothesis again (for \( N_G(H) \)), we get \( N_{G,n}(H) = N_{G,n-1}(N_G(H)) = G \). \( \square \)

3.4 Proposition. Suppose that \( G = \langle A, B \rangle \) and that \( G/Z_{H,n}(G) \) is a two element group for some \( n \geq 0 \). Then \( n = 0 \) and \( H \) is a normal subgroup of index 2 in \( G \).

Proof: Assume on the contrary, \( n \geq 1 \). With respect to 2.15, we can in fact assume that \( n = 1 \) and \( L_G(H) = 1 \). Then \( Z_{H,1}(G) = HZ(G) \) and \( H \cap Z(G) = 1 \).

By [1, Lemma 1.4], \( Z(G) \subseteq A \cap B \). Now, let \( a \in A \) and \( z \in Z(G) \). Then \( az = bu \) for some \( b \in A \) and \( u \in H \). We have \( u = b^{-1}az \) and \( c^{-1}uc = c^{-1}b^{-1}cb \cdot b^{-1}c^{-1}ac \cdot z = c^{-1}b^{-1}cb \cdot b^{-1}a^{-1}c^{-1}ac \in H \) for every \( c \in B \). This shows that \( u \in L_G(H) = 1 \) and \( az = bu \in A \). Now, since \( [G : HZ(G)] = 2 \), it is clear that \( A = Z(G) \cup az(G) \) for each \( a \in A - Z(G) \).

Finally, let \( a \in A \). Then \( a^{-1}b \notin H \) for some \( b \in B \) and, for every \( c \in B, c^{-1}a^{-1}bc = c^{-1}a^{-1}ca \cdot a^{-1}b \in H \). Thus \( a^{-1}b \in L_G(H) = 1 \) and \( a = b \in B \). We have proved that \( A = B \) and consequently \( G = \langle A, B \rangle = A \) is an abelian group, \( H = 1 \), \( Z_{H,1}(G) = G \) and \( G/Z_{H,1}(G) \) is trivial, a contradiction. \( \square \)

3.5 Lemma. Suppose that \( L_G(H) = 1 \), \( H \) is not abelian, every proper factor group of \( H \) is cyclic and that \( G = \langle A, B \rangle \). Then \( Z_{H,n}(G) \neq G \) for every \( n \geq 0 \), i.e., \( H \) is not subnormal in \( G \) (see 3.3).

Proof: Put \( N = Z_{H,1}(G) = L_G(HZ(G)) \), \( \overline{G} = G/N \) and \( \overline{H} = HN/N \overline{H} = H_1 \), \( H_1 = H \cap N \). If \( H_1 \neq 1 \), then \( \overline{H} \) is cyclic, and so \( \overline{A} = \overline{B} \) is an abelian subgroup of \( \overline{G} \) by [1, Corollary 2.3]. However, this implies that \( \overline{G} = \overline{A} \) is an abelian group, \( \overline{H} = 1 \), \( H \subseteq N = HZ(G) \) and \( H = H_1 \) is abelian by 2.10(iii), which is a contradiction.

We have proved that \( H_1 = 1 \), so that \( N = H_1Z(G) = Z(G) \) and \( \overline{H} \neq \overline{H} \).

Proceeding by induction, we get \( Z_{H,m}(G) = Z_m(G) \) for every \( m \geq 0 \). Now, if \( Z_{H,n}(G) = G \) for some \( n \geq 0 \), then \( G \) (and hence \( H \)) is nilpotent. But in such a case, \( Z(H) \neq 1 \), \( H/Z(H) \) is cyclic and this implies that \( H \) is abelian a contradiction. \( \square \)

3.6 Proposition. Suppose that every proper factorgroup of \( H \) is cyclic, that \( H \) is subnormal in \( G \) and that \( G = \langle A, B \rangle \). Then \( G' \subseteq N_G(H) \) and \( H \) is subnormal depth at most 2 in \( G \). Moreover, if \( H \) is not abelian, then \( G' \subseteq H \) and \( H \) is normal in \( G \).

Proof: First, assume that \( L_G(H) \neq 1 \). Then \( \overline{H} = H/L_G(H) \) is a cyclic subgroup of \( \overline{G} = G/L_G(H) \) and \( G' \subseteq H \) by [1, Theorem 2.2].
Next, let $L_G(H) = 1$. Then $H$ is abelian by 3.5 and if $H$ is cyclic, then we can use [1, Theorem 2.2] again to show that $H = 1$ and $G$ is abelian. Finally, if $H$ is not cyclic, than $H \cong \mathbb{Z}_p^2$ for a prime $p$ and the result follows from [5, Lemma 4.2]. □

3.7 Remark. According to [2], $G$ is soluble, provided that $G$ is finite and $H \cong S_3$. On the other hand, by 3.5, if $L_G(H) = 1$ and $G = \langle A, B \rangle$, then $H$ is not subnormal in $G$.

3.8 Proposition. Suppose that $L_G(H) = 1$ and $G$ is nilpotent of class at most 2. Then $[A, B] = 1$ and $A, B$ are isomorphic subgroups of $G$.

Proof: $[A, B] \subseteq H \cap G' \subseteq H \cap Z(G) \subseteq L_G(H) = 1$. The rest follows from [4, Lemma 2.3]. □

4. Examples

4.1. Let $G$ be the subgroup of $S_6$ (the symmetric group on {1,2, ...,6}) generated by the following permutations: (1 2), (3 4), (5 6), (1 3)(2 4), (1 3 5)(2 4 6). Further, let $K = \langle (1 2), (3 4), (5 6) \rangle \subseteq G$ and $H = \langle (1 2) \rangle \subseteq K$. Then $H$ is normal in $K$, $K$ is normal in $G$, card($G$) = 48, $K \cong \mathbb{Z}_2^3$, $H \cong \mathbb{Z}_2$, $L_G(H) = 1$ and $H$ is subnormal of depth 2 in $G$. On the other hand, $N_G(H) = \langle K, (3 5)(4 6) \rangle$, card($N_G(H)$) = 16, $K = L_G(N_G(H))$, $N_{G,2}(H) = N_G(N_G(H)) = N_G(H)$, $G/K \cong S_3$ and $Z(G) = 1$. Now, $Z_{H,n}(G) \neq G$ for every $n \geq 0$ and there exist no $H$-connected transversals to $H$ in $G$ (see 2.4(v) and 3.3).

4.2. Let $G$ be the subgroup of $S_{18}$ generated by $A = \{\text{id}, (1 2)(3 10 15 4 9 16)(5 12 17 6 11 18)(7 8)(13 14), (1 3 11 7 9 17 13 15 5)(2 10 18)(4 12 14)(6 8 16), (1 4 11 14 3 12 7 10 17 2 9 18 13 16 5 8 15 6), (1 5 10 14 6 9 7 11 16 2 12 15 13 17 4 8 18 3), (1 6 10 7 12 16 13 18 4)(2 11 15)(3 8 17)(5 9 14), (1 7 13)(2 8 14)(3 9 15)(4 10 16)(5 11 17)(6 12 18), (1 8 13 2 7 14)(3 16 9 4 15 10)(5 18 11 6 17 12), (1 9 5 7 15 11 13 3 17)(2 16 12)(4 18 8)(6 14 10), (1 10 5 14 9 6 7 16 11 2 15 12 13 4 17 8 3 18), (1 11 4 14 12 3 7 17 10 2 18 9 13 5 16 8 6 15), (1 12 4 7 18 10 13 6 16)(2 17 9)(3 19 11)(5 15 8), (1 13 7)(2 14 8)(3 15 9)(4 16 10)(5 17 11)(6 18 12), (1 14 7 2 13 8)(3 4)(5 6)(9 10)(11 12)(15 16)(17 18), (1 15 17 7 3 5 13 9 11)(2 4 6)(8 10 12)(14 16 18), (1 16 17 14 15 18 7 4 5 2 3 6 13 10 11 8 9 12), (1 17 16 14 18 15 7 5 4 2 6 3 13 11 10 8 12 9), (1 18 16 7 6 4 13 12 10)(2 5 3)(8 11 9)(14 17 15) \} and let $H$ be the stabilizer of 1 in $G$. Then $L_G(H) = 1$, card($H$) = 972 = $2^3 3^5$, $H$ is not nilpotent, $A$ is an $H$-selfconnected transversal to $H$ in $G = \langle A \rangle$, card($G$) = 17496 = $2^3 3^7$, and $Z_{H,3}(G) = G$ (cf. 2.13).

4.3. Let $G$ be the subgroup of $S_6$ generated by $A = \{\text{id}, (1 2)(3 4)(5 6), (1 3 5)(2 4 6), (1 4 5 2 3 6), (1 5 4 2 6 3), (1 6 4)(2 5 3) \}$ and let $H$ be the stabilizer of 1 in $G$. Then $L_G(H) = 1$, $H \cong \mathbb{Z}_2^2$, $A$ is an $H$-selfconnected transversal to $H$ in $G = \langle A \rangle$, card($G$) = 24, $Z_{H,2}(G) = G$, card($Z(G)$) = 2, $G$ is not nilpotent, card($N_G(H)$) = 8, $N_G(H) = HZ(G) = Z_{H,1}(G) \cong \mathbb{Z}_2^3$ and $G/Z_{H,1}(G) \cong \mathbb{Z}_3$ (cf. 2.4(iv) and 3.4).
4.4. Let $G$ be the subgroup of $S_6$ generated by $A = \{\text{id}, (1\ 2\ 3)(4\ 5\ 6), (1\ 3\ 2)(4\ 5\ 6), (1\ 4)(2\ 6\ 3\ 5), (1\ 5\ 3\ 6)(2\ 4), (1\ 6\ 2\ 5)(3\ 4)\}$ and let $H$ be the stabilizer of 1 in $G$. Then $L_G(H) = 1$, $H \cong S_3$ is soluble, $A$ is an $H$-selfconnected transversal to $H$ in $G = \langle A \rangle$, $\text{card}(G) = 36$, $G \neq Z_{H,n}(G)$ for every $n \leq 0$ and $H$ is not subnormal in $G$ (see 3.5).

4.5. Let $G$ be the subgroup of $S_4$ generated by $(1\ 2), (3\ 4), (1\ 3\ 2\ 4), (1\ 4\ 2\ 3)$, let $H$ be the stabilizer of 1 in $G$ and let $A = \{\text{id}, (1\ 2)(3\ 4), (1\ 3)(2\ 4), (1\ 4)(2\ 3)\}$. Then $L_G(H) = 1$, $H \cong Z_2$, $A$ is an $H$-selfconnected transversal to $H$ in $G$, $A \cong Z_2^{(2)}$ is a subgroup of $G$, $G$ is a dihedral eight-element group, $Z_{H,1}(G) \cong Z_2^{(2)}$ and $G/Z_{H,1}(G) \cong Z_2$ (cf. 3.4).

References


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