Existence of Absorbing Set for a Nonlinear Wave Equation *

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Abstract

We prove the existence of an absorbing set for a Cauchy problem involving a nonlinear wave equation.

Let \( u_0 \in W_2^{(1)}(0, l) \), \( u_1 \) and \( h \) are given elements in \( L_2(0, l) \), \( \alpha, \gamma \) and \( c \) be positive numbers and \( f(.) \in C^1(R) \) such that

\[
\mathcal{F}(s) = \int_0^s f(\eta)d\eta \geq -c, \quad (1)
\]

and

\[
f(s)s - \mathcal{F}(s) \geq -c \quad (2)
\]

for all \( s \in R \), where \( W_2^{(1)}(0, l) = \{ u : u, u' \in L_2(0, l), u(0) = u(l) = 0 \} \). We consider the following Cauchy problem:

\[
\begin{align*}
&u_{tt} - u_{xx} + \alpha u_t x + \gamma u_t + f(u) = h(x), \ x \in (0, l), \ t \in R^+ \\
u(x, 0) = u_0(x), \ u_t(x, 0) = u_1(x), \ x \in (0, l) \\
u(0, t) = u(l, t) = 0, \ t \in R^+.
\end{align*} \quad (3)
\]

A continuous semigroup of operators \( S(t) \) from \( X = W_2^{(1)}(0, l) \times L_2(0, l) \) into itself can be defined and satisfies the semigroup condition \( S(t+s) = S(t)S(s) \). We will prove the existence of an absorbing set in \( X \) by defining a Lyapunov like functional \( \phi(u, u_t) \). The methods used are inspired from the results of Ladyzhenskaya [3, 4] and used in Eden and Kalantarov [1, 2]. In the following we shall use the following notations:

\[
\|u\|^2 = \int_0^l u^2(x)dx,
\]

and

\[
(u, v) = \int_0^l u(x)v(x)dx.
\]

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We also use the Wirtinger inequality
\[ \int_0^l u^2(x)dx \leq \lambda \int_0^l u_x^2(x)dx \] (6)
where \( \lambda = l^2 / \pi^2 \).

**THEOREM 1.** Let \( u_0, u_1, h, \alpha, \gamma, c \) be given. Suppose \( f \in C^1(R) \) satisfies the conditions (1) and (2). Suppose further that the problem (3)-(5) has a unique weak solution. Then the semigroup \( S(t), t > 0 \), defined by \( S(t)(u_0, u_1) = (u(t), u_1(t)) \) generated by the problem (3)-(5) is bounded and dissipative.

**PROOF.** Suppose \( u(x, t) \) is a weak solution of (3)-(5). Let \( \delta > 0 \) be a positive number. Multiplying the equation (3) by \( u_t + \delta u \) and integrating over \((0, l)\) we get
\[
0 = \frac{d}{dt} \left( \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u_x\|^2 + (F(u), 1) - (h, u) + \delta (u, u_t) \right) \\
+ \frac{\delta \gamma}{2} \|u\|^2 + \gamma \|u_t\|^2 - \alpha \delta (u_x, u_t) + \delta \|u_x\|^2 \\
- \delta \|u_t\|^2 + \delta (f(u), u) - \delta (h, u).
\] (7)

We consider the functional
\[
\phi(u, u_t) = \frac{1}{2} \|u_t\|^2 + \frac{1}{2} \|u_x\|^2 + (F(u), 1) - (h, u) + \delta (u, u_t) + \frac{\delta \gamma}{2} \|u\|^2.
\] (8)

From (7) we write
\[
\frac{d}{dt} \phi(u, u_t) = -\gamma \|u_t\|^2 + \alpha \delta (u_x, u_t) - \delta \|u_x\|^2 + \delta \|u_t\|^2 - \delta (f(u), u) + \delta (h, u).
\] (9)

Let \( \eta > 0 \) be a positive number such that \( \eta < \delta \). Then from (7) we write
\[
\frac{d}{dt} \phi(u, u_t) + \eta \phi(u, u_t) = \frac{\eta}{2} \|u_t\|^2 + \frac{\eta}{2} \|u_x\|^2 + \eta (F(u), 1) - \eta (h, u) + \delta \eta (u, u_t) + \frac{\delta \eta \gamma}{2} \|u\|^2 \\
- \gamma \|u_t\|^2 + \alpha \delta (u_x, u_t) - \delta \|u_x\|^2 + \delta \|u_t\|^2 - \delta (f(u), u) + \delta (h, u).
\] (10)

By using (6) we obtain
\[
\delta \eta \|(u, u_t)\| \leq \frac{\delta \eta}{2} \|u_t\|^2 + \frac{\delta \eta l^2}{2\pi^2} \|u_x\|^2, \] (11)
\[
(\delta - \eta) \|(h, u)\| \leq \frac{\eta}{2} \|u_x\|^2 + \frac{l^2(\delta - \eta)^2}{2\pi^2} \|h\|^2, \] (12)

and
\[
\alpha \delta \|(u_x, u_t)\| \leq \frac{\gamma}{2} \|u_t\|^2 + \frac{\delta^2 \alpha^2}{2\gamma} \|u_x\|^2. \] (13)
With the help of the inequalities (11), (12) and (13) we obtain from (10) that
\[
\frac{d}{dt} \phi (u, u_t) + \eta \phi (u, u_t) \\
\leq \left( \delta - \frac{\gamma}{2} + \frac{\delta \gamma}{2} + \frac{\eta}{2} \right) \| u_t \|^2 + \left( \eta + \frac{\delta \gamma l^2}{2\pi^2} + \frac{\delta \eta \gamma}{2\pi^2} + \frac{\delta^2 \alpha^2}{2\gamma} - \delta \right) \| u_x \|^2 \\
- \delta ( (f(u), u) - (F(u), 1)) - (\delta - \eta) (F(u), 1) + \frac{l^2(\delta - \eta)^2}{2\eta\pi^2} \| h \|^2. \tag{14}
\]
Thus by choosing
\[
\delta < \min \left\{ \frac{\gamma}{4}, \frac{\gamma}{\alpha^2} \right\},
\eta < \min \left\{ \frac{2\gamma}{\gamma + 4}, \frac{\gamma^2}{2\pi^2\alpha^2 + \gamma l^2 + \gamma^2 l^2} \right\},
\]
and writing
\[
(\delta - \eta) (F(u), 1) \geq -c (\delta - \eta) I, \\
\delta ( (f(u), u) - (F(u), 1)) \geq -cbl,
\]
we get from (14) that
\[
\frac{d}{dt} \phi (u, u_t) + \eta \phi (u, u_t) \leq c_1 \tag{15}
\]
where
\[
c_1 = \frac{l^2(\delta - \eta)^2}{2\eta\pi^2} \| h \|^2 + cl (2\delta - \eta).
\]
It follows from (15) that
\[
\frac{d}{dt} (e^{\eta t} \phi (u, u_t)) \leq e^{\eta t} c_1 \tag{16}
\]
and
\[
\phi (u (., t), u_t (., t)) \leq \phi (u (., 0), u_t (., 0)) e^{-\eta t} + \frac{c_1}{\eta}. \tag{17}
\]
By using (8) and (1) and writing inequalities similar to (11) and (12), we may obtain
\[
\phi (u (., t), u_t (., t)) \geq \left( \frac{1}{2} - \frac{\delta \gamma l^2}{2\pi^2} + \frac{\delta^2}{2\pi^2} - \delta \right) \| u_t \|^2 \\
+ \left( \frac{1}{2} - \frac{\delta}{2} \right) \| u_t \|^2 - \left( cl + \frac{l^2}{2\delta\pi^2} \| h \|^2 \right). \tag{18}
\]
If we choose
\[
\delta < \min \left\{ 1, \frac{\pi^2}{l^2 \gamma + l^2 + \pi^2} \right\},
\]
then we get from (18) that
\[
\phi (u (., t), u_t (., t)) \geq a_0 \left( \| u_x \|^2 + \| u_t \|^2 \right) - c_2
\]
where

\[ a_0 = \frac{1}{2} \min \left\{ 1 - \frac{\delta l^2}{\pi^2} (\gamma + 1) - \delta, 1 - \delta \right\} \]

and

\[ c_2 = \frac{l^2}{2\delta \pi^2} \| h \|^2 + cl. \]

If we use this result in (17) we obtain

\[ \| u_x \|^2 + \| u_t \|^2 \leq e^{-\eta t} \phi (u (\cdot, 0), u_t (\cdot, 0)) + \frac{1}{a_0} \left( \frac{c_1}{\eta} + c_2 \right). \]

Then for \( R = \frac{1}{a_0} \left( \frac{c_1}{\eta} + c_2 \right) \), \( B = \{ (u, v) \in X : \|\{u, v\}\|_X \leq \sqrt{2R} \} \) is the absorbing set for the semigroup \( S(t) \) in \( X \). Thus the semigroup \( S(t), t > 0 \), is bounded and dissipative.

References


